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Integrable high-spin chain related to the elliptic solution of the Yang–Baxter equation

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Abstract. We discuss the higher-dimensional solutions of the Yang–Baxter equation and reflection equation in the elliptic case. The Hamiltonian of a spin-1 quantum anisotropic chain with non-trivial boundary term is given.

1. Introduction

Recently, increasing attention has been paid to quantum open chain systems [1–3]. This was initiated by Sklyanin to study a class of models with non-trivial boundary condition such as the 6-vertex model [1]. Manchenski and Nepomenchi [2] and Yue and Chen [3] developed this method to construct a great number of integrable models which have quantum group symmetry. The simplest example is the Heisenberg spin chain with fixed boundary term which has $SU_q(2)$ symmetry [4]. The Hamiltonian can be written in terms of the generator of $SU_q(2)$ in the fundamental representation. One open question is how to find the higher-spin chain with quantum symmetry. The standard method is the so-called fusion procedure [5–9]. Some examples have been given by Cherednik [5] and Manchenski and Nepomenchie [2] for the R-matrix and K-matrix, respectively, which are related to the trigonometric solution of the Yang–Baxter equation (YBE) [10]. These can be considered as the limit of the elliptic solutions of the YBE. So, it is important to study open higher-spin chain systems which exhibit Sklyanin algebra [11] and generalized algebra [12]. In this paper, we will discuss the anisotropic spin-1 chain system.

The programme of this paper is as follows. In section 2 we explicitly give an open spin-$\frac{1}{2}$ H_{xyz} model and review some well-known results which have been proposed by Sklyanin but without explicit expression. In section 3 we study the fusion of the R-matrix of an elliptic solution of the YBE [10], and discuss the invariance of the fused R-matrix. In section 4 we give the spin-1 K-matrix and show that it satisfies the spin-1 reflection equation. The Hamiltonian of the open anisotropic spin chain is given in section 5.

2. Open spin-$\frac{1}{2}$ chain

The R-matrix related to the 8-vertex model was first found by Baxter [13]. He has also set up the relation between the 8-vertex and the H_{xyz} model. In the notation of Faddeev and

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Takhatajan [14], the R-matrix can be written as

$$R(u) = \sum_{\alpha=1}^{4} w_\alpha(u)\sigma_\alpha \otimes \sigma_\alpha \tag{1}$$

where

$$\begin{aligned} w_1(u) + w_2(u) &= H(\eta)\Theta(u)\Theta(u+\eta) \\ w_1(u) - w_2(u) &= H(\eta)H(u)H(u+\eta) \\ w_3(u) + w_4(u) &= \Theta(\eta)\Theta(u)H(u+\eta) \\ w_4(u) - w_3(u) &= \Theta(\eta)H(u)\Theta(u+\eta) \end{aligned} \tag{2}$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

and $H(u)$ and $\Theta(u)$ are Jacobi theta functions. A more detailed definition is given in [14]. It is well known that the R-matrix satisfies the YBE

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{4}$$

In order to study open chain systems, Sklyanin has introduced two reflection equations for the special case. The generalized reflection equations are given in [2, 3]. For the 8-vertex model, the reflection equations are

$$R_{12}(u-v)\overset{1}{\mathcal{K}}_-(u)R_{21}(u+v)\overset{2}{\mathcal{K}}_-(v) = \overset{2}{\mathcal{K}}_-(v)R_{12}(u+v)\overset{1}{\mathcal{K}}_-(u)R_{21}(u-v) \tag{5}$$

and

$$R_{12}(-u+v)\overset{1}{\mathcal{K}}{}^{t_1}_+(u)R_{21}(-u-v-2\eta)\overset{2}{\mathcal{K}}{}^{t_2}_+(v) = \overset{2}{\mathcal{K}}{}^{t_2}_+(v)R_{12}(-u-v-2\eta)\overset{1}{\mathcal{K}}{}^{t_1}_+(u)R_{21}(-u+v) \tag{6}$$

where t_j stands for the transpose in jth space. Throughout the paper we use the notation $\overset{1}{A} = A \otimes 1$ and $\overset{2}{A} = 1 \otimes A$. These two matrix equations restrict the form of $\mathcal{K}_\pm$. Now we want to find the solution of the above equations. We assume that $\mathcal{K}_\pm(u)$ are diagonal matrices. Substituting the R-matrix defined by equation (1) and $\mathcal{K}_\pm$ into equations (5) and (6), one can find

$$\begin{aligned} \mathcal{K}_-(u) &= \mathrm{diag}(\mathrm{sn}(\xi + u - \eta/2), \mathrm{sn}(\xi - u + \eta/2)) \\ \mathcal{K}_+(u) &= \mathrm{diag}(\mathrm{sn}(\zeta - u - \eta), \mathrm{sn}(\zeta + u + \eta)) \end{aligned} \tag{7}$$

where ξ and ζ are arbitrary complex arguments. This solution was proposed by Cherednik and Sklyanin [1, 5], but the explicit form has not been given.

3. Fusion of the R-matrix

In this section we discuss the fusion procedure of the R-matrix, which was proposed by Cherednik *et al* [5–7]. First we consider the properties of the R-matrix given by equation (1). It is easy to show that

$$R_{12}(0) = \Theta(\eta)H(\eta)P_{12}$$
$$R_{12}(-\eta) = \Theta(\eta)H(-\eta)P_{12}^{-}. \tag{8}$$

Here P_{12} is a permutation operator and P_{12}^{-} is an antisymmetric projection operator, which satisfies

$$(P_{12}^{-})^2 = P_{12}^{-}$$
$$P_{12}^{-}A_{12}P_{12}^{-} = \mathrm{tr}_{12}(P_{12}^{-}A_{12})P_{12}^{-} \tag{9}$$

for $A_{12} \in V \otimes V$. Now, we use the fusion procedure to obtain a high-dimensional representation of the R-matrix. Although this idea was proposed by several authors, the explicit form of the R-matrix for the elliptic case has not been given. Taking $v = -\eta$ in equation (4), the YBE gives

$$R_{12}(u+\eta)R_{13}(u)R_{23}(-\eta) = R_{23}(-\eta)R_{13}(u)R_{12}(u+\eta). \tag{10}$$

Define

$$P_{12}^{+} = 1 - P_{12}^{-} \tag{11}$$

which has the following properties:

$$P_{12}^{+}P_{12}^{-} = 0 \qquad (P_{12}^{+})^2 = P_{12}^{+}. \tag{12}$$

It is obvious that the operator P_{12}^{+} is a symmetric projecting operator. Multiplying equation (10) by P_{12}^{+} from the right and the left, respectively, and using equation (12), we get

$$P_{23}^{-}R_{13}(u)R_{12}(u+\eta)P_{23}^{+} = 0 \tag{13}$$

$$P_{23}^{+}R_{12}(u+\eta)R_{13}(u)P_{23}^{-} = 0. \tag{14}$$

Define

$$R_{1(23)}(u) = P_{23}^{+}R_{13}(u-\eta)R_{12}(u)P_{23}^{+} \tag{15}$$

$$R'_{1(23)}(u) = P_{23}^{+}R_{12}(u+\eta)R_{13}(u)P_{23}^{+} \tag{16}$$

which respectively satisfy the YBE,

$$R_{12}(u-v)R_{1(34)}(u)R_{2(34)}(v) = R_{2(34)}(v)R_{1(34)}(u)R_{12}(u-v) \tag{17}$$

$$R_{12}(u-v)R'_{1(34)}(u)R'_{2(34)}(v) = R'_{2(34)}(v)R'_{1(34)}(u)R_{12}(u-v). \tag{18}$$

Here we only give the proof for $R(u)$:

$$\begin{aligned}
\text{LHS} &= R_{12}(u-v)P_{34}^{+}R_{14}(u-\eta)R_{13}(u)P_{34}^{+}P_{34}^{+}R_{24}(v-\eta)R_{23}(v)P_{34}^{+} \\
&= R_{12}(u-v)R_{14}(u-\eta)R_{13}(u)R_{24}(v-\eta)R_{23}(v)P_{34}^{+} \\
&= R_{24}(v-\eta)R_{14}(u-\eta)R_{12}(u-v)R_{13}(u)R_{23}(v)P_{34}^{+} \\
&= R_{24}(v-\eta)R_{14}(u-\eta)R_{23}(v)R_{13}(u)R_{12}(u-v)P_{34}^{+} \\
&= P_{34}^{+}R_{24}(v-\eta)R_{23}(v)P_{34}^{+}P_{34}^{+}R_{14}(u-\eta)R_{13}(u)P_{34}^{+}R_{12}(u-v) \\
&= \text{RHS}.
\end{aligned}$$

The proof for $R'(u)$ is similar. Substituting equation (1) into equation (15), we have

$$R_{1(34)}(u) = \begin{pmatrix} a' & 0 & i' & 0 & e' & 0 \\ 0 & b' & 0 & f' & 0 & h' \\ j' & 0 & c' & 0 & g' & 0 \\ 0 & g' & 0 & c' & 0 & j' \\ h' & 0 & f' & 0 & b' & 0 \\ 0 & e' & 0 & i' & 0 & a' \end{pmatrix} \tag{19}$$

where

$$\begin{aligned}
a' &= \mathrm{sn}(u+\eta)\,\mathrm{sn}(u) \qquad b' = \frac{\mathrm{sn}^2(u)[1-k^2\,\mathrm{sn}^4(\eta)]}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)} \\
c' &= \mathrm{sn}(u-\eta)\,\mathrm{sn}(u) \qquad i' = k\,\mathrm{sn}^2(\eta)\,\mathrm{sn}(u-\eta)\,\mathrm{sn}(u) \\
j' &= k\,\mathrm{sn}^2(\eta)\,\mathrm{sn}(u+\eta)\,\mathrm{sn}(u) \qquad e' = \frac{2k\,\mathrm{sn}^3(u)\,\mathrm{sn}(\eta)\mathrm{cn}(\eta)\,\mathrm{dn}(\eta)}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)} \\
f' &= \mathrm{sn}(u)\,\mathrm{sn}(\eta) \qquad g' = \frac{2k\,\mathrm{sn}(u)\,\mathrm{sn}(\eta)\,\mathrm{cn}(\eta)\,\mathrm{dn}(\eta)}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)} \\
h' &= k\,\mathrm{sn}(\eta)\,\mathrm{sn}(u+\eta)\,\mathrm{sn}(u)\,\mathrm{sn}(u-\eta).
\end{aligned} \tag{20}$$

Next, we consider the gauge invariance of the YBE. It is easy to show that the YBE is invariant under the transformation

$$\begin{aligned}
R_{12} &\to A_1B_2R_{12}A_1^{-1}B_2^{-1} \\
R_{13} &\to A_1C_3R_{13}A_1^{-1}C_3^{-1} \\
R_{23} &\to B_2C_3R_{23}B_2^{-1}C_3^{-1}
\end{aligned} \tag{21}$$

where A, B and C are non-degenerated matrices and belong to V^1, V^2 and V^3, respectively. For given $A \in V^1$, $B \in V^2$, $C \in V^{(34)}$, it is easy to show that the YBE (17) is invariant under the transformation

$$\begin{aligned}
R_{12} &\to A \otimes BR_{12}A^{-1} \otimes B^{-1} \\
R_{1(34)} &\to A \otimes CR_{1(34)}A^{-1} \otimes C^{-1} \\
R_{2(34)} &\to B \otimes CR_{2(34)}B^{-1} \otimes C^{-1}.
\end{aligned} \tag{22}$$

Taking $C = \mathrm{diag}(1, \sqrt{2\,\mathrm{cn}(\eta)\,\mathrm{dn}(\eta)}, 1)$ and $A = B = I_2$ and eliminating a scale $\mathrm{sn}(u)$ in the R-matrix, we get the convenient form

$$R_{1(34)}(u) = \begin{pmatrix} a & 0 & i & 0 & \rho h & 0 \\ 0 & b & 0 & \rho & 0 & \rho g \\ j & 0 & c & 0 & \rho f & 0 \\ 0 & \rho f & 0 & c & 0 & j \\ \rho g & 0 & \rho & 0 & b & 0 \\ 0 & \rho h & 0 & i & 0 & a \end{pmatrix} \tag{23}$$

where

$$\begin{aligned} &\rho = \sqrt{2\,\mathrm{sn}^2(\eta)\,\mathrm{cn}(\eta)\,\mathrm{dn}(\eta)} \\ &a = \mathrm{sn}(u+\eta) \qquad c = \mathrm{sn}(u-\eta) \\ &i = k\,\mathrm{sn}^2(\eta)\,\mathrm{sn}(u-\eta) \qquad j = k\,\mathrm{sn}^2(\eta)\,\mathrm{sn}(u+\eta) \\ &b = \frac{\mathrm{sn}(u)(1-k^2\,\mathrm{sn}^4(\eta))}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)} \qquad f = \frac{1}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)} \\ &g = k\,\mathrm{sn}(u+\eta)\,\mathrm{sn}(u-\eta) \qquad h = \frac{k\,\mathrm{sn}^2(u)}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)}. \end{aligned} \tag{24}$$

In order to compare equation (23) with three-dimensional trigonometric R-matrix, we investigate the degenerate case of equation (24). Taking the τ-argument in the theta function as approaching $\mathrm{i}\infty$, we get $k \to 0$, $\mathrm{sn}(u) \to \mathrm{sh}(u)$, $\mathrm{cn}(u) \to \mathrm{ch}(u)$ and $\mathrm{dn}(u) \to 1$. So, equation (24) changes into

$$R_{1(34)}(u) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & d & 0 & 0 \\ 0 & 0 & c & 0 & d & 0 \\ 0 & d & 0 & c & 0 & 0 \\ 0 & 0 & d & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \tag{25}$$

where

$$\begin{aligned} &a = \mathrm{sh}(u+\eta) \qquad b = \mathrm{sh}(u) \qquad c = \mathrm{sh}(u-\eta) \\ &d = \sqrt{2\,\mathrm{sh}^2(\eta)\,\mathrm{ch}(\eta)}. \end{aligned} \tag{26}$$

This is coincident with the result given by Cherednik [5]. In a similar way we can define

$$R_{(12)(34)}(u) = P_{12}^{+} R_{1(34)}(u) R_{2(34)}(u+\eta) P_{12}^{+} \tag{27}$$

which acts on $V^{(12)} \otimes V^{(34)}$ with $\dim V^{(12)} = \dim V^{(34)} = 3$. This R-matrix satisfies the YBE

$$R_{(12)(34)}(u-v) R_{(12)(56)}(u) R_{(34)(56)}(v) = R_{(34)(56)}(v) R_{(12)(56)}(u) R_{(12)(34)}(u-v). \tag{28}$$

The proof is as follows:

$$\begin{aligned}
\mathrm{LHS} = {} & R_{14}(u-v-\eta)R_{13}(u-v)R_{24}(u-v)R_{23}(u-v+\eta)R_{1(56)}(u)R_{2(56)}(u+\eta) \\
& \times R_{3(56)}(v)R_{4(56)}(v+\eta)P_{34}^{+}P_{12}^{+} \\
= {} & R_{14}(u-v-\eta)R_{24}(u-v)R_{13}(u-v)R_{1(56)}(u)R_{3(56)}(v)R_{2(56)}(u+\eta)R_{23}(u-v+\eta) \\
& \times R_{4(56)}(v+\eta)P_{34}^{+}P_{12}^{+} \\
= {} & R_{3(56)}(v)R_{14}(u-v-\eta)R_{1(56)}(u)R_{13}(u-v)R_{24}(u-v)R_{2(56)}(u+\eta)R_{4(56)}(v+\eta) \\
& \times R_{23}(u-v+\eta)P_{34}^{+}P_{12}^{+} \\
= {} & R_{3(56)}(v)R_{4(56)}(v+\eta)R_{1(56)}(u)R_{2(56)}(u+\eta)R_{14}(u-v-\eta)R_{13}(u-v)R_{24}(u-v) \\
& \times R_{23}(u-v+\eta)P_{34}^{+}P_{12}^{+} \\
= {} & \mathrm{RHS}.
\end{aligned}$$

In the above calculation, we have repeatedly used equation (17) and rearrange the R-matrices appropriately. Substituting equation (23) into equation (27), we get

$$R_{(12)(34)}(u) = \begin{pmatrix}
A_{11} & 0 & A_{13} & 0 & C_{12} & 0 & B_{11} & 0 & B_{13} \\
0 & A_{22} & 0 & C_{21} & 0 & C_{23} & 0 & B_{22} & 0 \\
A_{31} & 0 & A_{33} & 0 & C_{32} & 0 & B_{31} & 0 & B_{33} \\
0 & D_{12} & 0 & E_{11} & 0 & E_{13} & 0 & D_{32} & 0 \\
D_{21} & 0 & D_{23} & 0 & E_{22} & 0 & D_{23} & 0 & D_{21} \\
0 & D_{32} & 0 & E_{31} & 0 & E_{33} & 0 & D_{12} & 0 \\
B_{33} & 0 & B_{31} & 0 & C_{32} & 0 & A_{33} & 0 & A_{31} \\
0 & B_{22} & 0 & C_{23} & 0 & C_{21} & 0 & A_{22} & 0 \\
B_{13} & 0 & B_{11} & 0 & C_{12} & 0 & A_{13} & 0 & A_{11}
\end{pmatrix} \tag{29}$$

where

$$A_{11} = \mathrm{sn}(u+2\eta)[\mathrm{sn}(u+\eta) + k^2\,\mathrm{sn}^4(\eta)\,\mathrm{sn}(u-\eta)]$$

$$A_{13} = k\,\mathrm{sn}(u)\,\mathrm{sn}^2(\eta)[\mathrm{sn}(u+\eta) + \mathrm{sn}(u-\eta)]$$

$$A_{31} = k\,\mathrm{sn}(u+\eta)\,\mathrm{sn}^2(\eta)[\mathrm{sn}(u+\eta) + \mathrm{sn}(u-\eta)]$$

$$A_{33} = \mathrm{sn}(u)[\mathrm{sn}(u-\eta) + k^2\,\mathrm{sn}^4(\eta)\,\mathrm{sn}(u+\eta)]$$

$$A_{22} = \frac{\mathrm{sn}(u)\,\mathrm{sn}(u+\eta)[1-k^2\,\mathrm{sn}^4(\eta)]^2}{[1-k^2\,\mathrm{sn}^2(u+\eta)\,\mathrm{sn}^2(\eta)][1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)]}$$

$$B_{11} = \frac{\rho^2 k^2\,\mathrm{sn}^2(u)}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)} \qquad B_{13} = \frac{\rho^2 k^2\,\mathrm{sn}^3(u)\,\mathrm{sn}(u+\eta)}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)}$$

$$B_{22} = \frac{\rho^2 k\,\mathrm{sn}(u+\eta)[\mathrm{sn}(u-\eta)+\mathrm{sn}(u+\eta)]}{1-k^2\,\mathrm{sn}^2(u+\eta)\,\mathrm{sn}^2(\eta)}$$

$$B_{31} = \frac{\rho^2}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)} \qquad B_{33} = \frac{\rho^2 k\,\mathrm{sn}(u)\,\mathrm{sn}(u+2\eta)}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(\eta)}$$

$$C_{12} = \frac{\rho k[\operatorname{sn}^3(u+\eta) + \operatorname{sn}^2(\eta)\operatorname{sn}(u-\eta)]}{[1-k^2\operatorname{sn}^2(u+\eta)\operatorname{sn}^2(\eta)]} + \frac{\rho k \operatorname{sn}(u+\eta)\operatorname{sn}^2(u)(1-k^2\operatorname{sn}^4(\eta))}{[1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(\eta)][1-k^2\operatorname{sn}^2(u+\eta)\operatorname{sn}^2(\eta)]}$$

$$C_{21} = \frac{2\rho \operatorname{sn}(u+\eta)\operatorname{cn}(\eta)\operatorname{dn}(\eta)[1-k^2\operatorname{sn}^4(\eta)]}{[1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(\eta)][1-k^2\operatorname{sn}^2(u+\eta)\operatorname{sn}^2(\eta)]}$$

$$C_{23} = \rho k \operatorname{sn}(u)[(\operatorname{sn}(u+\eta)\operatorname{sn}(u-\eta) + \operatorname{sn}^2(\eta)] + \frac{\rho k(1-k^2\operatorname{sn}^4(\eta))\operatorname{sn}(u+2\eta)\operatorname{sn}^2(u)}{[1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(\eta)]}$$

$$C_{32} = \rho\frac{[1+k^2\operatorname{sn}^2(u+\eta)\operatorname{sn}^2(\eta)][\operatorname{sn}(u+\eta)+\operatorname{sn}(u-\eta)]}{[1-k^2\operatorname{sn}^2(u+\eta)\operatorname{sn}^2(\eta)]}$$

$$D_{12} = \frac{\rho \operatorname{sn}(u+\eta)[1-k^2\operatorname{sn}^4(\eta)]}{1-k^2\operatorname{sn}^2(u+\eta)\operatorname{sn}^2(\eta)}$$

$$D_{21} = \frac{\rho k \operatorname{sn}^2(u)\operatorname{sn}(u+2\eta)[1-k^2\operatorname{sn}^4(\eta)]}{1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(\eta)}$$

$$D_{23} = \frac{\rho \operatorname{sn}(u)[1-k^2\operatorname{sn}^4(\eta)]}{1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(\eta)}$$

$$D_{32} = \frac{\rho k \operatorname{sn}^2(u)\operatorname{sn}(u+\eta)[1-k^2\operatorname{sn}^4(\eta)]}{[1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(\eta)][1-k^2\operatorname{sn}^2(u+\eta)\operatorname{sn}^2(\eta)]}$$

$$2E_{11} = k^2\operatorname{sn}^4(\eta)\operatorname{sn}(u-\eta)\operatorname{sn}(u) + k^2\operatorname{sn}^4(\eta)\operatorname{sn}(u+\eta)\operatorname{sn}(u+2\eta) + \operatorname{sn}(u+\eta)\operatorname{sn}(u) + \operatorname{sn}(u-\eta)\operatorname{sn}(u+2\eta) + \frac{2\operatorname{sn}^2(\eta)\operatorname{cn}(\eta)\operatorname{dn}(\eta)[1+k^2\operatorname{sn}^3(u)\operatorname{sn}(u+2\eta)]}{1-k^2\operatorname{sn}^2(\eta)\operatorname{sn}^2(u)}$$

$$E_{33} = E_{11} \qquad E_{13} = E_{31}$$

$$E_{22} = \frac{\operatorname{sn}(u)\operatorname{sn}(u+\eta)[1-k^2\operatorname{sn}^4(\eta)]^2}{[1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(\eta)][1-k^2\operatorname{sn}^2(u+\eta)\operatorname{sn}^2(\eta)]} + \frac{\rho^2[1+k^2\operatorname{sn}^3(u+\eta)\operatorname{sn}(u-\eta)]}{1-k^2\operatorname{sn}^2(u+\eta)\operatorname{sn}^2(\eta)}$$

$$E_{13} = \frac{2\rho^2 k \operatorname{sn}(u)\operatorname{sn}(u+\eta)\operatorname{cn}(\eta)\operatorname{dn}(\eta)}{[1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(\eta)][1-k^2\operatorname{sn}^2(u+\eta)\operatorname{sn}^2(\eta)]}. \tag{30}$$

Similarly, we consider the gauge transformation $C \otimes I_3$, where I_3 is a three-dimensional identity operator and C is the same as before. Under the transformation, the elements of R change into

$$C_{ij} \to \frac{C_{ij}}{\sqrt{2\operatorname{cn}(\eta)\operatorname{dn}(\eta)}} \qquad D_{ij} \to D_{ij}\sqrt{2\operatorname{cn}(\eta)\operatorname{dn}(\eta)} \tag{31}$$

and the other elements remain invariant. In fact, the transformations (22) and (31) can be considered together. This gives a symmetric transformation,

$$R'(u) = C \otimes CR(u)C^{-1} \otimes C^{-1}. \tag{32}$$

After some calculations, one finds that

$$R'_{(12)(34)}(u)|_{u=0} = P_{(12)(34)} \tag{33}$$

where $P_{(12)(34)}$ is a three-dimensional permutation operator. From equation (28) we know that R' satisfies the YBE. In section 5, we will simply rewrite R' as R.

4. Fusion of the K matrix

This section will contribute to the fusion of the K-matrix. Taking $v = u+\eta$ in equation (5), one can find

$$P_{12}^{-}\overset{1}{\mathcal{K}}_{-}(u)R_{21}(2u+\eta)\overset{2}{\mathcal{K}}_{-}(u+\eta) = \overset{2}{\mathcal{K}}_{-}(u+\eta)R_{12}(2u+\eta)\overset{1}{\mathcal{K}}_{-}(u)P_{21}^{-} \quad (34)$$

which means that

$$P_{12}^{-}\overset{1}{\mathcal{K}}_{-}(u)R_{21}(2u+\eta)\overset{2}{\mathcal{K}}_{-}(u+\eta)P_{21}^{+} = 0. \quad (35)$$

Define

$$\mathcal{K}_{(12)} = P_{12}^{+}\overset{1}{\mathcal{K}}_{-}(u-\eta/2)R_{12}(2u)\overset{2}{\mathcal{K}}_{-}(u+\eta/2)P_{21}^{+} \quad (36)$$

which satisfies the generalized reflection equation

$$\begin{aligned} &R_{(12)(34)}(u-v)\overset{(12)}{\mathcal{K}}_{-}(u)\tilde{R}_{(34)(12)}(u+v)\overset{(34)}{\mathcal{K}}_{-}(v) \\ &\quad = \overset{(34)}{\mathcal{K}}_{-}(v)\tilde{R}_{(12)(34)}(u+v)\overset{(12)}{\mathcal{K}}_{-}(u)R_{(34)(12)}(u-v) \end{aligned} \quad (37)$$

where

$$\tilde{R}_{(34)(12)}(u) = P_{12}^{+}P_{34}^{+}R_{31}(u-\eta)R_{41}(u)R_{32}(u)R_{42}(u+\eta)P_{12}^{+}P_{34}^{+}.$$

It is worthwhile pointing out that the reflection equations satisfied by the fused R-matrix and K-matrix are different from the original ones. This is due to the property of the fused R-matrix. Now we return to prove equation (37);

$$\begin{aligned} \text{LHS} &= P_{12}^{+}P_{34}^{+}R_{14}(u-v-\eta)R_{24}(u-v)R_{13}(u-v)R_{23}(u-v+\eta)\overset{1}{\mathcal{K}}_{-}(u-\eta/2)R_{12}(2u) \\ &\quad \times \overset{2}{\mathcal{K}}_{-}(u+\eta/2)R_{31}(u+v-\eta)R_{32}(u+v)R_{41}(u+v)R_{42}(u+v+\eta) \\ &\quad \times \overset{3}{\mathcal{K}}_{-}(u-\eta/2)R_{34}(2v)\overset{4}{\mathcal{K}}_{-}(v+\eta/2)P_{12}^{+}P_{34}^{+} \\ &= P_{12}^{+}P_{34}^{+}R_{14}(u-v-\eta)R_{24}(u-v)R_{31}(u-v)\overset{1}{\mathcal{K}}_{-}(u-\eta/2)R_{13}(u+v-\eta)R_{12}(2u) \\ &\quad \times \overset{3}{\mathcal{K}}_{-}(u-\eta/2)R_{23}(u+v)\overset{2}{\mathcal{K}}_{-}(u+\eta/2)R_{32}(u-v+\eta)R_{41}(u+v)R_{42}(u+v+\eta) \\ &\quad \times R_{34}(2v)\overset{4}{\mathcal{K}}_{-}(v+\eta/2)P_{12}^{+}P_{34}^{+} \\ &= P_{12}^{+}P_{34}^{+}\overset{3}{\mathcal{K}}_{-}(u-\eta/2)R_{34}(2v)R_{13}(u+v-\eta)R_{14}(u-v-\eta)\overset{1}{\mathcal{K}}_{-}(u-\eta/2)R_{23}(u+v) \\ &\quad \times R_{41}(u+v)R_{12}(2u)\overset{4}{\mathcal{K}}_{-}(v+\eta/2)R_{42}(u+v+\eta)\overset{2}{\mathcal{K}}_{-}(u+\eta/2)R_{24}(u-v) \\ &\quad \times R_{31}(u-v)R_{32}(u-v+\eta)P_{12}^{+}P_{34}^{+} \\ &= P_{12}^{+}P_{34}^{+}\overset{3}{\mathcal{K}}_{-}(u-\eta/2)R_{34}(2v)\overset{4}{\mathcal{K}}_{-}(v+\eta/2)R_{13}(u+v-\eta)R_{23}(u+v)R_{41}(u+v) \\ &\quad \times R_{42}(u+v+\eta)\mathcal{K}_{-}^{1}(u-\eta/2)R_{12}(2u)\overset{2}{\mathcal{K}}_{-}(u+\eta/2)R_{14}(u-v-\eta)R_{24}(u-v) \\ &\quad \times R_{13}(u-v)R_{23}(u-v+\eta)P_{12}^{+}P_{34}^{+} \\ &= \text{RHS}. \end{aligned}$$

Besides, the $\mathcal{K}_{(12)}$ matrix has the following property:

$$\mathcal{K}_{(12)}(0) = sn(\xi - \eta)\, sn(\xi)\, sn(\eta) P^{+}_{12}. \tag{38}$$

Substituting equation (7) into equation (36), we get

$$\mathcal{K}_{(12)}(u) = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix} \tag{39}$$

where

$$\begin{aligned}
K_{11} &= \frac{\mathrm{sn}(u+\xi)\,\mathrm{sn}(\xi+u-\eta)}{\mathrm{sn}(\xi)\,\mathrm{sn}(\xi-\eta)} \\
K_{13} &= \frac{k\,\mathrm{sn}(2u)\,\mathrm{sn}(\eta)\,\mathrm{sn}(\xi-u-\eta)\,\mathrm{sn}(\xi+u-\eta)}{\mathrm{sn}(\xi)\,\mathrm{sn}(\xi-\eta)} \\
K_{31} &= \frac{k\,\mathrm{sn}(2u)\,\mathrm{sn}(\eta)\,\mathrm{sn}(\xi-u)\,\mathrm{sn}\,\xi+u)}{\mathrm{sn}(\xi)\,\mathrm{sn}(\xi-\eta)} \\
K_{33} &= \frac{sn(\xi-u)\,sn(\xi-u-\eta)}{\mathrm{sn}(\xi)\,\mathrm{sn}(\xi-\eta)}
\end{aligned} \tag{40}$$

$$\begin{aligned}
K_{22} = {} & \frac{\mathrm{sn}(2u)[\mathrm{sn}(\xi+u-\eta)\,\mathrm{sn}(\xi-u-\eta)+\mathrm{sn}(\xi-u)\,\mathrm{sn}(\xi+u)]}{2\,\mathrm{sn}(2u+\eta)\,\mathrm{sn}(\xi)\,\mathrm{sn}(\xi-\eta)} \\
& + \frac{\mathrm{sn}(\eta)[\mathrm{sn}(\xi+u-\eta)\,\mathrm{sn}(\xi+u)+\mathrm{sn}(\xi-u-\eta)\,\mathrm{sn}(\xi-u)]}{2\,\mathrm{sn}(2u+\eta)\,\mathrm{sn}(\xi)\,\mathrm{sn}(\xi-\eta)}.
\end{aligned} \tag{41}$$

Here, we have renormalized $\mathcal{K}_{-}(u)$ so that $\mathcal{K}_{-}(0) = 1$.

Similarly, the solution of the modified reflection equation can be satisfied by $\mathcal{K}_{+}$. However, we do not need to solve directly the equation due to the existence of algebraic automorphism with maps $\mathcal{K}_{-}$ into $\mathcal{K}_{+}$ [1, 3]:

$$\phi : \mathcal{K}_{-}(u) \to \mathcal{K}_{+}(u) = \mathcal{K}^{t}_{-}(-u - \eta/2). \tag{42}$$

Generally, repeating the same procedure, one can find the arbitrary dimensional R-matrix and K-matrix. Here we do not need it for the spin-1 chain systems. In a future paper we will discuss the arbitrary dimensional K-matrix for the generalized 8-vertex model—the Z_n symmetric model.

5. The Hamiltonian of spin-1 systems

In this section, we study the Hamiltonian of anisotropic spin-1 systems with closed and open boundary conditions. We rewrite the $R_{(12)(34)}$ as R_{ij}, $(i, j = 0, 1, \ldots)$ which acts on $V^i \otimes V^j$, and $\dim V^i = \dim V^j = 3$.

(a) Periodic case. We define the transfer matrix as

$$T_N(u) = R_{0N}(u) R_{0N-1}(u) \ldots R_{01}(u). \tag{43}$$

It is easy to show that the monodromy matrix $\tau(u) = \mathrm{tr}_0\, T_N(u)$ consists of a commutative family, i.e.

$$[\tau(u), \tau(v)] = 0. \tag{44}$$

The proof is similar to one given by Sklyanin [1]. The expression of $\tau(u)$ about argument u gives an infinite number of conservation quantities, which means the system is integrable. Differentiating $\log \tau(u)$ with respect to u at $u = 0$, we get the Hamiltonian

$$H = \sum_{i=1}^{N} H_{ii+1} \tag{45}$$

where

$$H_{ii+1} = \begin{pmatrix} A'_{11} & 0 & 0 & 0 & C'_{12} & 0 & 0 & 0 & 0 \\ 0 & D'_{12} & 0 & E'_{11} & 0 & A'_{31} & 0 & 0 & 0 \\ A'_{31} & 0 & 0 & 0 & C'_{32} & 0 & A'_{33} & 0 & A'_{31} \\ 0 & E'_{11} & 0 & C'_{21} & 0 & C'_{23} & 0 & A'_{31} & 0 \\ D'_{21} & 0 & D'_{23} & 0 & E'_{22} & 0 & D'_{23} & 0 & D'_{21} \\ 0 & A'_{31} & 0 & C'_{23} & 0 & C'_{21} & 0 & E'_{11} & 0 \\ A'_{31} & 0 & A'_{33} & 0 & C'_{32} & 0 & 0 & 0 & A'_{31} \\ 0 & 0 & 0 & A'_{31} & 0 & E'_{11} & 0 & D'_{12} & 0 \\ 0 & 0 & 0 & 0 & C'_{12} & 0 & 0 & 0 & A'_{11} \end{pmatrix} \tag{46}$$

where

$$\begin{aligned} A'_{11} &= [1 - k^2 \mathrm{sn}^2(2\eta)\, \mathrm{sn}^2(\eta)][\mathrm{sn}(3\eta) + k^2 \mathrm{sn}^5(\eta)] \qquad A'_{31} = k\rho^2 \mathrm{sn}(2\eta) \\ A'_{33} &= -\frac{\rho^2}{\mathrm{sn}(2\eta)} \qquad C'_{12} = \frac{2k\,\mathrm{sn}(\eta)\rho^2}{1 - k^2 \mathrm{sn}^4(\eta)} \\ C'_{21} &= \frac{\rho^2 \mathrm{sn}(2\eta)[1 + k^2 \mathrm{sn}^4(\eta)]}{2\,\mathrm{sn}^2(\eta)} \qquad C'_{32} = \frac{\rho^2[1 + k^2 \mathrm{sn}^4(\eta)]}{1 - k^2 \mathrm{sn}^4(\eta)} \\ D'_{12} &= \frac{\rho^2 \mathrm{sn}(2\eta)[1 + k^2 \mathrm{sn}^4(\eta)]}{2\,\mathrm{sn}^2(\eta)} \qquad D'_{23} = \frac{\rho^2[1 - k^2 \mathrm{sn}^4(\eta)]}{\mathrm{sn}(\eta)} \\ E'_{11} &= \mathrm{sn}(\eta)[1 - k^2 \mathrm{sn}^4(\eta)] \qquad E'_{22} = \frac{\rho^2}{\mathrm{sn}^2(\eta)}. \end{aligned} \tag{47}$$

Formally, this Hamiltonian is not Hermitian. But the nearest-neighbour interaction depends upon two complex arguments η and τ in the definition of the theta function. One can get a Hermitian Hamiltonian by choosing these arguments properly. Besides, our result under the trigonometric limit is coincident with the known one given by Cherednik [5].

(b) Open case. For an open chain, we define the transfer matrix as

$$\tau(u) = \mathrm{tr}_0\, \mathcal{K}_+(u) T_N(u) \mathcal{K}_-(u) T_N^{-1}(-u). \tag{48}$$

Following the procedure given in [3], one can show that

$$[\tau(u), \tau(v)] = 0 \tag{49}$$

which comprises a commutative family. This means that the model is integrable. Differentiating $\tau(u)$ with respect to u at $u = 0$, one finds

$$H_{\text{open}} = \sum_{i=1}^{N-1} H_{ii+1} + b_0 + b_N \tag{50}$$

where

$$b_0 = \begin{pmatrix} K'_{11} & 0 & 0 \\ 0 & K'_{22} & 0 \\ 0 & 0 & K'_{33} \end{pmatrix} \tag{51}$$

where

$$\begin{aligned} K'_{11} &= \frac{\operatorname{sn}(2\xi - \eta)[1 - k^2 \operatorname{sn}^2(\xi - \eta)\operatorname{sn}^2(\xi)]}{\operatorname{sn}(\xi)\operatorname{sn}(\xi - \eta)} \\ K'_{33} &= -K'_{11} \\ K'_{22} &= \frac{\operatorname{sn}(\eta)}{\operatorname{sn}(\xi)\operatorname{sn}(\xi - \eta)}[1 + k^2 \operatorname{sn}^2(\xi - \eta)\operatorname{sn}^2(\xi)] \end{aligned} \tag{52}$$

and

$$b_N = \begin{pmatrix} K^+_{11} & 0 & K^+_{13} \\ 0 & K^+_{22} & 0 \\ K^+_{31} & 0 & K^+_{33} \end{pmatrix} \tag{53}$$

where

$$\begin{aligned} K^+_{11} &= \frac{\operatorname{sn}(2\zeta)[1 - k^2 \operatorname{sn}^2(\zeta - \eta/2)\operatorname{sn}^2(\zeta + \eta/2)]}{\operatorname{sn}(\zeta)\operatorname{sn}(\zeta - \eta)} \\ K^+_{33} &= \frac{\operatorname{sn}(2\zeta - \eta)[1 - k^2 \operatorname{sn}^2(\zeta - \eta/2)\operatorname{sn}^2(\zeta - 3\eta/2)]}{\operatorname{sn}(\zeta)\operatorname{sn}(\zeta - \eta)} \\ K^+_{13} &= \frac{-k \operatorname{sn}^2(\eta)[1 - k^2 \operatorname{sn}^2(\zeta - \eta/2)\operatorname{sn}^2(\zeta + \eta/2)]}{\operatorname{sn}(\zeta)\operatorname{sn}(\zeta - \eta)} \\ K^+_{31} &= \frac{-k \operatorname{sn}^2(\eta)[1 - k^2 \operatorname{sn}^2(\zeta - \eta/2)\operatorname{sn}^2(\zeta - 3\eta/2)]}{\operatorname{sn}(\zeta)\operatorname{sn}(\zeta - \eta)} \\ K^+_{22} &= \frac{\operatorname{sn}(\eta)[1 - k^2 \operatorname{sn}^2(2\eta)\operatorname{sn}^2 \eta)]\operatorname{sn}(\zeta - \eta/2)[\operatorname{sn}(\zeta - 3\eta/2) + \operatorname{sn}(\zeta + \eta/2)]}{\operatorname{sn}^2(2\eta)\operatorname{sn}(\zeta - \eta)\operatorname{sn}(\zeta)} \\ &\quad + \frac{\operatorname{sn}(\eta)[K^+_{13} + K^+_{31}]}{2k \operatorname{sn}(\eta)\operatorname{sn}(2\eta)} + \frac{\operatorname{sn}(\eta)[K^+_{11} + K^+_{33}]}{2\operatorname{sn}(\eta)}. \end{aligned} \tag{54}$$

So, this equation defines the open anisotropic spin-1 chain system. One can check it by taking its trigonometric limit, which is coincident with the known results [9]. This model may be solved by using the quantum inverse scattering method (QISM) [1, 14, 15] as in the trigonometric case in which the key is the commutative relation. We do not solve this model here but only give the operator relations

$$R^{a_1c_1}_{a_2c_2}(u - v)T^{c_1d_1}_{-}(u)\tilde{R}^{d_1b_1}_{c_2d_2}(u + v)T^{d_2b_2}_{-}(v) = T^{a_2c_2}_{-}(v)\tilde{R}^{a_1c_1}_{c_2d_2}(u + v)T^{c_1d_1}_{-}(u)R^{d_1b_1}_{d_2b_2}(u - v) \tag{55}$$

where the double index denotes summation.

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